Corrigendum: Regional Data in Macroeconomics

Gabriel Chodorow-Reich
Harvard University

James Stratton
Harvard University

January 2023

Summary. This note corrects a small error in Example II of Chodorow-Reich (2020).[1] Example II considers government spending multipliers in an economy with multiple regions, each containing "Old Keynesian" rule-of-thumb agents. There is an error in the expression for fiscal multipliers in the example. This note corrects the error and provides a derivation for the corrected expression. The asymptotic results for the impacts of local expenditure are correctly stated in the article.

Correction. Equation (12) of Chodorow-Reich (2020) describes element (i, j) of the matrix $\mathbf{B}$ as:

$$b_{i,j} = \begin{cases} \frac{1}{m} + \frac{1}{Nm - \frac{N-1}{\alpha\rho}m^2}, & i = j, \\ \frac{1}{Nm - \frac{N-1}{\alpha\rho}m^2}, & i \neq j. \end{cases}$$

The second term in the $i = j$ case, and the full expression in the $i \neq j$, case, have been reversed in sign. The corrected equation is:

$$b_{i,j} = \begin{cases} \frac{1}{m} + \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm}, & i = j, \\ \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm}, & i \neq j. \end{cases}$$

Derivation. The model is described in Chodorow-Reich (2020). For convenience, the key parts are excerpted below:

> The economy again consists of N regions, each with fixed size $1/N$ (no inter-regional migration, unlike the example above). Let $\mathbf{c} = (c_1, \ldots, c_N)'$ denote the vector of consumption expenditures in each region, $\mathbf{y} = (y_1, \ldots, y_N)'$ the vector of outputs, and $\mathbf{g} = (g_1, \ldots, g_N)'$ the vector of government purchases, where each variable c_i, y_i, g_i is the (level) deviation from its steady state value. A representative agent in each region allocates $1 - \alpha$ of her expenditure to locally-produced output and $\frac{\alpha}{N-1}$ of her expenditure to output produced in each other region. Market clearing then requires:
>
> $$\mathbf{y} = \mathbf{A}\mathbf{c} + \mathbf{g},$$
>
> $$\text{where:} \qquad \mathbf{A} = \begin{pmatrix} (1-\alpha) & \frac{\alpha}{N-1} & \cdots & \frac{\alpha}{N-1} \\ \frac{\alpha}{N-1} & (1-\alpha) & \cdots & \frac{\alpha}{N-1} \\ \vdots & \frac{\alpha}{N-1} & \ddots & \vdots \\ \frac{\alpha}{N-1} & \cdots & \frac{\alpha}{N-1} & (1-\alpha) \end{pmatrix}.$$
>
> The agent also has a marginal propensity to consume out of income of ρ:
>
> $$\mathbf{c} = \rho\mathbf{y}.$$

Our goal is to find the effects of local government spending on local output, output in other regions, and aggregate output. That is (following the notation in Chodorow-Reich (2020)), we look for a matrix $\mathbf{B}$ such that $\mathbf{y} = \mathbf{B}\mathbf{g}$

[1] Chodorow-Reich, Gabriel (2020). "Regional Data in Macroeconomics: Some Advice for Practicioners." Journal of Economic Dynamics and Control 115: 103875.

for an arbitrary $\mathbf{g}$ vector:

$$\mathbf{y} = \mathbf{A}\mathbf{c} + \mathbf{g} = \mathbf{B}\mathbf{g}.$$
$$\Rightarrow \rho\mathbf{A}\mathbf{y} + \mathbf{g} = \mathbf{B}\mathbf{g}.$$
$$\Rightarrow \rho\mathbf{A}\mathbf{B}\mathbf{g} + \mathbf{g} = \mathbf{B}\mathbf{g}.$$
$$\Rightarrow (I - \rho\mathbf{A})\mathbf{B} = I.$$
$$\Rightarrow \mathbf{B} = (I - \rho\mathbf{A})^{-1}.$$

The last line appears in Chodorow-Reich (2020), but there is an error in the matrix inversion. Using the definition of $\mathbf{A}$, we write:

$$\mathbf{B} = \begin{pmatrix} 1-\rho(1-\alpha) & -\frac{\alpha\rho}{N-1} & \cdots & -\frac{\alpha\rho}{N-1} \\ -\frac{\alpha\rho}{N-1} & 1-\rho(1-\alpha) & \cdots & -\frac{\alpha\rho}{N-1} \\ \vdots & -\frac{\alpha\rho}{N-1} & \ddots & \vdots \\ -\frac{\alpha\rho}{N-1} & \cdots & -\frac{\alpha\rho}{N-1} & 1-\rho(1-\alpha) \end{pmatrix}^{-1}.$$

In order to compute this inverse, we use the following fact: if an $L \times L$ matrix $\mathbf{D}$ can be written $\mathbf{D} = (a-b)I + b\mathbf{J}$, where a and b are scalars with $a \neq b$ and $a - b + Lb \neq 0$, I is the $L \times L$ identity matrix, and $\mathbf{J}$ is an $L \times L$ matrix of ones, then the diagonal elements of $\mathbf{D}^{-1}$ are equal to $\frac{1}{a-b} - \frac{b}{(a-b)(a-b+Lb)}$ and the off-diagonal elements are equal to $\frac{-b}{(a-b)(a-b+Lb)}$.[2] Writing $\mathbf{B}^{-1}$ as $\Big(1 - \rho(1-\alpha) + \frac{\alpha\rho}{N-1})\Big)I - \frac{\alpha\rho}{N-1}\mathbf{J}$ and applying this result,[3] the diagonal elements of $\mathbf{B}$ are:

$$b^d = \frac{1}{1-\rho(1-\alpha)+\frac{\alpha\rho}{N-1}} + \frac{\frac{\alpha\rho}{N-1}}{\Big(1-\rho(1-\alpha)+\frac{\alpha\rho}{N-1}\Big)\Big(1-\rho(1-\alpha)+\frac{\alpha\rho}{N-1}-\frac{N}{N-1}\alpha\rho\Big)}.$$

Defining $m \equiv 1 - \rho(1-\alpha) + \frac{\alpha\rho}{N-1}$, we then have:

$$\begin{aligned} b^d &= \frac{1}{m} + \frac{\frac{\alpha\rho}{N-1}}{m(m - \frac{N}{N-1}\alpha\rho)} \\ &= \frac{1}{m} + \frac{1}{m(\frac{m}{\alpha\rho}(N-1) - N)} \\ &= \frac{1}{m} + \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm}. \end{aligned}$$

The off-diagonal elements of $\mathbf{B}$ are:

$$b^{od} = \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm}.$$

Then we conclude that element (i,j) of $\mathbf{B}$ is:

$$b_{i,j} = \begin{cases} \frac{1}{m} + \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm}, & i = j, \\ \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm}, & i \neq j. \end{cases}$$

Implications for multipliers as the number of regions $N \to \infty$. Chodorow-Reich (2020) makes several statements regarding the behavior of the multipliers in $\mathbf{B}$ as $N \to \infty$. Each of these statements is true with the

[2] To derive this result, we can first conjecture that D^{-1} can be written $xI + y\mathbf{J}$, for some scalars x and y. Under the conjecture, we have $[(a-b)I + \mathbf{J}][xI + y\mathbf{J}] = I$, which implies $x(a-b)I + [y(a-b+Lb) + xb]\mathbf{J} = I$. After making the assumptions that $a \neq b$ and $a - b + Lb \neq 0$, we can simultaneously solve $x(a-b) = 1$ and $y(a-b+Lb) + xb = 0$ to yield $x = \frac{1}{a-b}$ and $y = \frac{-b}{(a-b)(a-b+Lb)}$. Since x and y are scalar constants (given a, b, and L), this verifies the conjecture. Finally, since $\mathbf{D}^{-1} = xI + y\mathbf{J}$, its diagonal elements are equal to $x + y = \frac{1}{a-b} - \frac{b}{(a-b)(a-b+Lb)}$, and its off-diagonal elements are equal to $y = \frac{-b}{(a-b)(a-b+Lb)}$.

[3] We can apply the result because the on-diagonal and on-diagonal elements differ, since $(1-\rho) > 0 > -\frac{\alpha\rho}{N-1}$, and $1 - \rho(1-\alpha) - (N-1)\frac{\alpha\rho}{N-1} = 1 - \rho > 0$.

corrected expression. First, as $N \to \infty$, the off-diagonal elements of $\mathbf{B}$ converge to zero:

$$\begin{aligned} b^{od} &= \frac{1}{\frac{N-1}{\alpha\rho}m^2 - Nm} \\ &= \frac{1}{Nm\left(\frac{N-1}{N}\frac{m}{\alpha\rho} - 1\right)} \\ &= \frac{N\alpha\rho}{Nm(N-1)(1-\rho)} \\ &= \frac{\alpha\rho}{m(N-1)(1-\rho)} \\ &\to^{N\to\infty} 0. \end{aligned}$$

Second, as $N \to \infty$, the diagonal elements of $\mathbf{B}$ converge to $\frac{1}{1-\rho(1-\alpha)}$:

$$\begin{aligned} b^d &= \frac{1}{1-\rho(1-\alpha)+\frac{\alpha\rho}{N-1}} + b^{od} \\ &\to^{N\to\infty} \frac{1}{1-\rho(1-\alpha)}. \end{aligned}$$

Third, the column sum down $\mathbf{B}$ – the impact on the aggregate economy of local expenditure – is:

$$\begin{aligned} b^d + (N-1)b^{od} &= \frac{1}{m} + N\frac{\alpha\rho}{m(N-1)(1-\rho)}. \\ &= \frac{1}{m}\frac{1}{1-\rho}\left(1-\rho+\frac{N}{N-1}\alpha\rho\right). \\ &= \frac{1}{1-\rho}\frac{1-\rho+\frac{N}{N-1}\alpha\rho}{1-\rho+\alpha\rho+\frac{\alpha\rho}{N-1}} \\ &= \frac{1}{1-\rho}, \end{aligned}$$

which demonstrates that the impact of local expenditure on the aggregate economy not only converges to the "Old Keynesian" closed economy multiplier as $N \to \infty$, but in fact is equal to the "Old Keynesian" mutliplier for any $N \geq 2$.